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# Renormalisation-group treatment of systems with superconducting and other orderings in a magnetic field

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**Abstract.** The scaling behaviour of a superconducting system with another ordering in a magnetic field is considered by the renormalisation-group approach. Exact recursion relations up to the order $O(\epsilon)$ are obtained and analysed. The usual bicritical and tetracritical fixed points do not appear for physically interesting values of the symmetry indices of the order parameters. This phenomenon is associated with the strong influence of the vector-potential fluctuations on the critical behaviour of the system.

## 1. Introduction

The critical behaviour of systems with two interacting order parameters in the framework of the generalised Ginzburg–Landau–Wilson models has been studied intensively in the mean-field approximation (e.g. Imry 1975) and by the renormalisation-group (RG) approach (e.g. Kosterlitz *et al* 1976, Lyuksyutov *et al* 1975, Gorodetsky and Zaprudsky 1975; see also Aharony 1976).

Systems with a fluctuating order parameter coupled to a gauge field, namely a superconductor in a magnetic field (Halperin *et al* 1974) and the transition between nematic and smectic-A liquid crystal mesophases (Halperin and Lubensky 1974; see also Chen *et al* 1978) have also been considered. The fluctuations of the gauge field (the vector potential of the magnetic field and the direction vector in the liquid crystal) change the universality at the transition point to (i) a new Halperin–Lubensky–Ma-(HLM) type second-order phase transition for $n > n_c = 365{\cdot}9$ and (ii) a 'weak' first-order phase transition for $n < n_c$ (see Halperin *et al* 1974), where $n$ is the symmetry index of the order parameter.

Recently, the RG approach has been applied by Grewe and Schuh (1979) to the problem of co-existence of superconductivity and ferromagnetism in a magnetic field, using the free energy functional proposed by Blount and Varma (1979). In the vicinity of the ferromagnetic–superconducting phase boundary, the critical fluctuations of the magnetic ordering are absorbed into the fluctuations of the vector potential. Thus one obtains a quantitative modification of the HLM recursion relations. This result is due to the fact that the magnetic field is conjugated to the magnetisation. Hence a term linear in the magnetic order parameter appears to be relevant for the result of the RG treatment.

In this paper we apply the RG approach (Wilson and Kogut 1974) to systems in a magnetic field which contain two interacting order parameters, namely a superconducting and another (non-magnetic) one. Systems with two such order parameters are, for instance, a superconductor with a structural distortion associated with doubling of the lattice periodicity (Mattis and Langer 1970, Brankov and Tonchev 1976) or a two-band semi-metal with both exitonic and superconducting phase transitions (Rusinov *et al* 1973, Kopayev and Molotkov 1979).

## 2. The model

We start from a free energy functional of the form

$$\mathscr{F}\{\psi, \phi, \boldsymbol{A}\} = -\int \mathrm{d}\boldsymbol{x}\{a|\psi(\boldsymbol{x})|^2 + \gamma_\psi|(\nabla - \mathrm{i}q_0\boldsymbol{A}(\boldsymbol{x}))\psi(\boldsymbol{x})|^2 + \tfrac{1}{2}b_\psi|\psi(\boldsymbol{x})|^4 + \frac{1}{8\pi\mu}(\mathrm{rot}\,\boldsymbol{A}(\boldsymbol{x}))^2 + \tfrac{1}{2}r\phi^2(\boldsymbol{x}) + \tfrac{1}{2}\gamma_\phi(\nabla\phi(\boldsymbol{x}))^2 + \tfrac{1}{4}b_\phi\phi^4(\boldsymbol{x}) + \tfrac{1}{2}c\phi^2(\boldsymbol{x})|\psi(\boldsymbol{x})|^2 \tag{1}$$

where $\psi(\boldsymbol{x})$ is the superconducting order parameter, $\boldsymbol{A}(\boldsymbol{x})$ is the vector potential and $\phi(\boldsymbol{x})$ is the other (non-magnetic) order parameter. As usual, $a = a'(T - T_\psi)/T_\psi$, $r = r'(T - T_\phi)/T_\phi$, $\mu$ is the magnetic permeability of the system, $q_0 = 2e$ is twice the electron charge and $b_\psi$, $b_\phi$, $c$, $\gamma_\psi$ and $\gamma_\phi$ are assumed to be analytic functions of the temperature (including the critical points $T_\psi$ and $T_\phi$, $T_\psi \simeq T_\phi$). Here $\hbar = c = k_\mathrm{B} = 1$. The dimension of the space is $d = 4 - \epsilon$. The Coulomb gauge $\mathrm{div}\,\boldsymbol{A}(\boldsymbol{x}) = 0$ is assumed. There are several characteristic lengths in model (1), but the quantitative results of the RG treatment do not change if we use a common cut-off for the short-length fluctuations of the fields $\sigma = \psi, \phi, \boldsymbol{A}$.

In momentum space we shall work, for instance, with the dimensionless wavevectors $\boldsymbol{q} = \boldsymbol{q}/q_\mathrm{c}$ ($q_\mathrm{c}$ is the momentum cut-off). After some simple transformations of the fields, the free energy (1) in the momentum-space representation is

$$\mathscr{F} = \mathscr{F}_1 + \mathscr{F}_2 + \mathscr{F}_3 + \mathscr{F}_4 + \mathscr{F}_5 + \mathscr{F}_6 \tag{2}$$

where

$$\mathscr{F}_1 = -\sum_{\boldsymbol{q}}\left(\sum_\alpha (r_\psi + \boldsymbol{q}^2)\psi_\alpha^*(\boldsymbol{q})\psi_\alpha(\boldsymbol{q}) + \tfrac{1}{2}\sum_N (r_\phi + \boldsymbol{q}^2)\phi_N(\boldsymbol{q})\phi_N(-\boldsymbol{q}) + \frac{1}{8\pi\mu}\sum_i \boldsymbol{q}^2 A_i(\boldsymbol{q})A_i(-\boldsymbol{q})\right) \tag{3}$$

$$\mathscr{F}_2 = -\frac{q_0}{\Omega^{1/2}}\sum_{i\alpha;\boldsymbol{q}_1\boldsymbol{q}_2}(\boldsymbol{q}_1 + \boldsymbol{q}_2)_i A_i(\boldsymbol{q}_1 - \boldsymbol{q}_2)\psi_\alpha^*(\boldsymbol{q}_1)\psi_\alpha(\boldsymbol{q}_2) \tag{4}$$

$$\mathscr{F}_3 = -\frac{q_0^2}{\Omega}\sum_{i\alpha;\boldsymbol{q}_1\ldots\boldsymbol{q}_3} A_i(\boldsymbol{q}_1)A_i(\boldsymbol{q}_2)\psi_\alpha^*(\boldsymbol{q}_3)\psi_\alpha(\boldsymbol{q}_3 - \boldsymbol{q}_1 - \boldsymbol{q}_2) \tag{5}$$

$$\mathscr{F}_4 = -\frac{v_0}{4\Omega}\sum_{N,L;\boldsymbol{q}_1\ldots\boldsymbol{q}_3}\phi_N(\boldsymbol{q}_1)\phi_N(\boldsymbol{q}_2)\phi_L(\boldsymbol{q}_3)\phi_L(-\boldsymbol{q}_1 - \boldsymbol{q}_2 - \boldsymbol{q}_3) \tag{6}$$

$$\mathscr{F}_5 = -\frac{w_0}{2\Omega}\sum_{\alpha,N;\boldsymbol{q}_1\ldots\boldsymbol{q}_3}\phi_N(\boldsymbol{q}_1)\phi_N(\boldsymbol{q}_2)\psi_\alpha^*(\boldsymbol{q}_3)\psi_\alpha(\boldsymbol{q}_3 - \boldsymbol{q}_1 - \boldsymbol{q}_2) \tag{7}$$

$$\mathscr{F}_6 = -\frac{u_0}{2\Omega}\sum_{\alpha\beta;\boldsymbol{q}_1\ldots\boldsymbol{q}_3}\psi_\alpha^*(\boldsymbol{q}_1)\psi_\beta^*(\boldsymbol{q}_2)\psi_\alpha(\boldsymbol{q}_3)\psi_\beta(\boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{q}_3). \tag{8}$$

In (3)–(8) $\Omega$ is a dimensionless volume $\Omega = Vq_c^d$ and

$$r_\psi = a/\gamma_\psi q_c^2 \qquad r_\phi = r/\gamma_\phi q_c^2$$

$$u_0 = \frac{b_\psi}{\gamma_\psi^2} q_c^{d-4} \qquad v_0 = \frac{b_\phi}{\gamma_\phi^2} q_c^{d-4} \qquad w_0 = \frac{c}{\gamma_\psi \gamma_\phi} q_c^{d-4}.$$

The order parameters $\psi(\boldsymbol{q})$ and $\phi(\boldsymbol{q})$ are generalised to a $(n/2)$-component complex field and to a $m$-component real field, respectively. The vector potential $\boldsymbol{A}(\boldsymbol{q})$ is a $d$-dimensional vector. The suffixes $\alpha$, $N$ and $i$ denote the components of the corresponding fields. The Feynman graph rules for model (2) are standard. The free correlation functions

$$G^{(0)}_{\sigma\alpha\alpha'}(\boldsymbol{q}, \boldsymbol{q}') = \langle \sigma^*_\alpha(\boldsymbol{q}) \sigma_{\alpha'}(\boldsymbol{q}')\rangle_0 \qquad \sigma \equiv \psi, \phi, \boldsymbol{A}$$

are

$$G^{(0)}_{\sigma\alpha\alpha'}(\boldsymbol{q}, \boldsymbol{q}') = \delta_{\alpha\alpha'} \delta(\boldsymbol{q} - \boldsymbol{q}') \frac{1}{r_\sigma + q^2} \qquad \sigma \equiv \psi, \phi$$

and

$$G^{(0)}_{A\alpha\alpha'}(\boldsymbol{q}, \boldsymbol{q}) = \delta(\boldsymbol{q} - \boldsymbol{q}') \left( \delta_{\alpha\alpha'} - \frac{q_\alpha q'_\alpha}{\boldsymbol{q}^2} \right) \frac{4\pi\mu}{\boldsymbol{q}^2}.$$

## 3. The renormalisation-group transformation

Exact recursion relations to order $O(\epsilon)$ for the free energy (2) are obtained through a straightforward application of the RG approach (Wilson and Kogut 1974). We overcome some complications caused by the inconsistency between the RG procedure and the requirement that the model be gauge invariant using the method proposed by Halperin *et al* (1974). The partial trace of the probability distribution $\exp\{\mathscr{F}\}$, taken over the high-momentum ($\exp(-s) < q < 1$, $0 < s < \infty$) degrees of freedom $\sigma_\alpha(\boldsymbol{q})$, is calculated to first order in the vertex constants for the parameters $r_\psi$ and $r_\phi$, and to second order for the vertex constants $u_0$, $v_0$, $w_0$, $q_0$ and $q_0^2$. The contributions to the $\boldsymbol{q}$-dependent terms in the correlation functions $G_\psi$ and $G_A$ are also accounted for to second order in the perturbation expansion. Note that the actual small parameters appear to be $q_0^2$, $u_0$, $v_0$ and $w_0$. They are assumed to be of order $\epsilon = 4 - d$. Then, one must check the perturbation expansion up to fourth order because of the term $\mathscr{F}_2$ (equation (4)) linear in the effective charge $q_0$. This yields the result that only contributions to second order in the perturbation expansion are relevant (for details of the calculations see Chen *et al* (1978) and Grewe and Schuh (1979)). The RG procedure is accomplished by the rescaling $\boldsymbol{q} = \boldsymbol{q}' \exp(-s)$ and $\sigma'(\boldsymbol{q}') = \exp[s(1 - \frac{1}{2}\eta_\sigma)]\sigma(\boldsymbol{q})$, where $\eta_\sigma$ are the anomalous dimensions of the fields $\sigma \equiv \psi, \phi, \boldsymbol{A}$. The recursion relations to order $O(\epsilon)$ corresponding to the free energy (1) are

$$r'_\psi = \exp[s(2 - \eta_\psi)][r_\psi + \tfrac{1}{2}(n+2) f(s, r_\psi) u_0 + 12\pi f(s, 0) q_0^2 \mu + \tfrac{1}{2} m f(s, r_\phi) w_0] \tag{9}$$

$$r'_\phi = \exp[s(2 - \eta_\phi)][r_\phi + (m+2) f(s, r_\phi) v_0 + \tfrac{1}{2} n f(s, r_\psi) w_0] \tag{10}$$

$$u'_0 = \exp[s(\epsilon - 2\eta_\psi)][u_0 - \tfrac{1}{2}(n+8) g(s) u_0^2 - \tfrac{1}{2} m g(s) w_0^2 - 96\pi^2 q_0^4 \mu^2 g(s)] \tag{11}$$

$$v'_0 = \exp[s(\epsilon - 2\eta_\phi)][v_0 - (m+8) g(s) v_0^2 - \tfrac{1}{4} n g(s) w_0^2] \tag{12}$$

$$w_0' = \exp[s(\epsilon - \eta_\psi - \eta_\phi)][w_0 - 2g(s)w_0^2 - \tfrac{1}{2}(n+2)g(s)u_0w_0 - (m+2)g(s)v_0w_0] \tag{13}$$

$$(\mu')^{-1} = \exp(-s\eta_A)\mu^{-1}\left(1 + \frac{ns}{12\pi}q_0^2\mu\right) \tag{14}$$

$$q_0' = \exp[\tfrac{1}{2}(\epsilon - \eta_A)s]q_0 \tag{15}$$

$$1 = \exp(-s\eta_\psi)\left(1 - \frac{3s}{2\pi}q_0^2\mu\right) \tag{16}$$

where

$$f(s, r_\sigma) = \frac{1}{8\pi^2}\left(\frac{1-\exp(-2s)}{2} - r_\sigma s\right) \qquad g(s) = \frac{s}{8\pi^2}.$$

Recursion relations (9)–(16) are a direct generalisation of those for two coupled fields (Kosterlitz *et al* 1976) and those for a superconductor in a magnetic field (Halperin *et al* 1974). Up to order $O(\epsilon)$ we have the anomalous dimensions $\eta_\phi$, $\eta_\phi = 0$.

The assumption $\eta_A \neq \epsilon$ in equation (15) would lead to $q_0^* = \infty$ (if $\eta_A < \epsilon$) or to $q_0^* = 0$ (if $\eta_A > \epsilon$) at the fixed point. The first case does not give any finite fixed points. In the case $q_0^* = 0$, the free energy differs from the original one (2) because vertices of type $\mathscr{F}_2$ and $\mathscr{F}_3$ would be absent in the fixed-point free energy.

Denoting $t = q_0^2\mu/12\pi\epsilon$, we obtain for $\eta_\psi$ and $\eta_A$ (see (14) and (16))

$$\eta_\psi = -18\epsilon t^* \tag{17}$$

$$\eta_A = n\epsilon t^*. \tag{18}$$

With the anomalous dimensions $\eta_A$, $\eta_\psi$ and $\eta_\phi = 0$ we can study the recursion relations in a reduced parameter space $\boldsymbol{\mu} = (r_\psi, r_\phi, u, v, w, t)$, where

$$u = u_0/8\bar{\epsilon} \qquad v = v_0/4\bar{\epsilon} \qquad w = w_0/8\bar{\epsilon} \qquad (\bar{\epsilon} = 8\pi^2\epsilon).$$

Then we obtain from (9)–(16):

$$r_\psi' = \exp(2s)(1 + 18\epsilon st)[r_\psi + 4(n+2)\bar{\epsilon}f(s, r_\psi)u + 4m\bar{\epsilon}f(s, r_\psi)w + 18\bar{\epsilon}f(s, 0)t] \tag{19}$$

$$r_\phi' = \exp(2s)[r_\phi + 4(m+2)\bar{\epsilon}f(s, r_\phi)v + 4n\bar{\epsilon}f(s, r_\psi)w] \tag{20}$$

$$u' = \exp(\epsilon s)(1 + 36\epsilon st)[u - 4(n+8)\epsilon su^2 - 4m\epsilon sw^2 - 27\epsilon st^2] \tag{21}$$

$$v' = \exp(\epsilon s)[v - 4(m+8)\epsilon sv^2 - 4n\epsilon sw^2] \tag{22}$$

$$w' = \exp(\epsilon s)(1 + 18\epsilon st)[w - 16\epsilon sw^2 - 4(n+2)\epsilon suw - 4(m+2)\epsilon svw] \tag{23}$$

$$t' = \exp(\epsilon s)t(1 - ns\epsilon t). \tag{24}$$

## 4. Analysis of the fixed points

From (24) one obtains two types of fixed points $\boldsymbol{\mu}^* = (r_\psi^*, \ldots)$ corresponding to $t^* = 0$ and $t^* = 1/n$. In both cases the fixed-point values $u^*$, $v^*$ and $w^*$ are (see (21)–(23))

$$(1 + 36t^*)u^* = 4(n+8)u^{*2} + 4mw^{*2} + 27t^{*2} \tag{25}$$

$$v^* = 4(m+8)v^{*2} + 4nw^{*2} \tag{26}$$

$$(1 + 18t^*)w^* = 16w^{*2} + 4(n+2)u^*w^* + 4(m+2)v^*w^*. \tag{27}$$

If the values $u^*$, $v^*$ and $w^*$ are known, $r_\psi^*$ and $r_\Phi^*$ are to be

$$r_\psi^* = -2\epsilon[(n+2)u^* + mw^* + \tfrac{9}{2}t^*]$$
$$r_\Phi^* = -2\epsilon[(m+2)v^* + nw^*]. \qquad (28)$$

The eigenvalues $\exp\{sy_{\mu_i}\}$ ($\mu_i = r_\psi, \ldots$) of the linearised transformation matrix for the relevant variables which determine the scaling behaviour of the system are (see the appendix)

$$\begin{aligned} y_t &= -\epsilon \qquad \text{for } t^* = 1/n \\ y_t &= \epsilon \qquad \text{for } t^* = 0 \end{aligned} \qquad (29)$$

and

$$y_{r_\psi, r_\Phi} = 2 + \epsilon(9t^* - 2(n+2)u^* - 2(m+2)v^* \pm \{[9t^* + 2(m+2)v^* - 2(n+2)u^*]^2 + 16mnw^{*2}\}^{1/2}). \qquad (30)$$

The eigenvalues $y_u$, $y_v$ and $y_w$ could be found from a third-order algebraic equation (see equations (A.7) and (A.8)). Using these values of $y_u$, $y_v$ and $y_w$ one might establish the stability of every fixed point and the corrections to the critical behaviours.

We shall comment briefly on the following cases.

### 4.1. $t^* = 0$

From (17) and (18) we have $\eta_\psi = \eta_A = 0$. This is the case $q_0^* = 0$ discussed above. The fixed points (25)–(27) and the critical exponents (29)–(30) (despite the presence of the parameter $t$ in the recursion relations) up to order $O(\epsilon)$ are the same as those described by Kosterlitz *et al* (1976). According to (29), in a magnetic field the fixed points of a system with two ordering parameters are unstable with respect to perturbations of the magnetic field around the value $H = 0$.

### 4.2. $t^* = 1/n$

*4.2.1. Decoupled behaviour: $w^* = 0$.* In this case for $n > n_c = 365{\cdot}9$ one always obtains an unstable Gaussian–HLM fixed point and a Heisenberg–HLM fixed point for $m \neq -8$ and $n > n_c$. The last one is stable (with respect to $w$-type fluctuations) if $n$ and $m$ satisfy the condition

$$32 - nm - 2n - 2m + \tfrac{1}{2}(m+8)\left(n + 2 + \frac{216}{n} - \frac{n+2}{n}(n^2 - 360n - 2160)^{1/2}\right) + O(\epsilon) < 0. \qquad (31)$$

In the particular case when in the original model $u_0 = v_0$, the Gaussian–HLM fixed point vanishes, whereas the Heisenberg–HLM fixed point is possible only if

$$108(m+8)^2 - n(n+36)(m+8) + n^2(n+8) = 0 \qquad n > 365{\cdot}9. \qquad (32)$$

The critical exponents for $n$ and $m$ satisfying (32) are the same for fixed points as different as the Heisenberg and HLM ones.

Here we shall point out the following interesting behaviour of the HLM fixed point when the term $27t^{*2}$ is removed from equation (25) for $u^*$. This is possible when the symmetry index $n$ is very large. Then, instead of the HLM fixed point, we get a

Wilson-like fixed point

$$r_{\psi}^{*} = -\frac{n+2}{2(n+8)}\left(1+\frac{36}{n}\right)\epsilon - \frac{9\epsilon}{n}$$
$$u^{*} = \frac{1+(36/n)}{4(n+8)} \tag{33}$$

and, of course, a 'Gaussian' one $(r_{\psi}^{*}, u^{*}, t^{*}) = (0, 0, 1/n)$. The critical exponents corresponding to the fixed point (33) are

$$y_{r_{\psi}} = 2+\epsilon\left[\frac{18}{n} - \frac{n+2}{n+8}\left(1+\frac{36}{n}\right)\right]$$

$$y_{u} = -\left(1+\frac{36}{n}\right)\epsilon$$

where $t^{*}$ and $y_t$ are given by (29). For $t^{*}=0$ one obtains the usual results (Wilson and Kogut 1974). The origin of the term $27t^{*2}$ in (25) is due to the presence of the vertex $\mathscr{F}_3$ in (2). For large $n$ its removal breaks down the gauge invariance of model (2).

*4.2.2. Coupled behaviour $w^{*} \neq 0$.* The presence of terms with $t^{*}=1/n$ in (25)–(27) reflects the absence of a 'bicritical' fixed-point solution of type $u^{*}=v^{*}=w^{*}$. The term $4mw^{*2}$ in the equation for $u^{*}$ (25) modifies the critical value of the symmetry index $n$ from $n_c$ to $n_c' > n_c$ for $m>0$, and to $n_c' < n_c$ for $m<0$. When $m=0$, the system (25)–(27) decouples and the solutions for $u^{*}$, $v^{*}$ and $w^{*}$ can be determined analytically. We have another analytic solution of the system (25)–(27) for $n \to \infty$. Then, as seen from (31), the physical system falls into the range of stability of the decoupled fixed points for $m > -2$.

## 5. Discussion

We have presented the RG recursion relations for a system containing two order parameters with an interaction of type $\phi^2\psi^2$. Moreover, one of the order parameters ($\psi$), with a charge $q_0$, is coupled to a magnetic field. For such a system, exact recursion relations are found which generalise the recursion relations for two important cases: (i) for a superconductor in a magnetic field (Halperin *et al* 1974) and (ii) for a system with two coupled order parameters (Kosterlitz *et al* 1976; see also Lyuksyutov *et al* 1975).

Our consideration demonstrates that the RG recursion relations (19)–(24) do not possess any stable fixed points for the physically interesting values of the symmetry index $n$. This result provides an additional example of systems where the so called 'weak' first-order phase transitions (Halperin *et al* 1974) occur. When one of the order parameters ($\psi$) is charged, the critical behaviour predicted by Kosterlitz *et al* (1976), even in zero magnetic field, must be changed drastically due to fluctuations of the magnetic field. The consequences are: (i) there is no tetracritical behaviour, i.e. no intersection points of two second-order phase boundary lines on the phase diagram of the system exist; (ii) owing to the absence of the bicritical fixed point, the system has no points (on the phase diagram) where a first-order transition line would branch into two second-order ones. Thus, if a mixed $\psi$–$\phi$ phase occurs, it would not be bounded by second-order lines only. For instance, possible simple fragments on the phase diagram allowing the presence of the mixed $\psi$–$\phi$ phase are shown in figure 1. These features are consequences of the fact that the vector potential makes some weak first-order transition lines (those connected with the $\psi$ ordering).

For systems with two order parameters, an effective extension of the superconducting critical region is possible due to the influence of the other ordering $\phi(x)$ near the point $T_\psi = T_\phi$ (Hornreich and Schuster 1979). Then one might suggest that the range of the weak first-order transition is also extended near this point.

Stable coupled fixed points should be looked for when $n$ and $m$ satisfy the inverse inequality (31). Then one has to find the real roots of an algebraic equation of fourth order in $w^*$ with coefficients which are polynomials of $n$ and $m$. In the limiting case $n \to \infty$ the effects of the vector-potential fluctuations are negligible.

The results mentioned above obtained for the example of superconductivity are applicable to every system of two ordering parameters where one of them is coupled to a gauge field.


**Figure 1.** Possible fragments on the phase diagram of the system. The full curves represent the first-order phase transition, the broken ones those of second order.

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## Appendix

In this appendix the evaluation of the critical exponents is outlined. Let us denote the parameters in (19)–(24) by $\boldsymbol{\mu} = \{\mu_1, \ldots, \mu_6\} = \{r_\psi, \ldots, t\}$. The critical exponents $y_{\mu_i}$ corresponding to a fixed point are obtained from the eigenvalues $\lambda_{\mu_i} = \exp\{s y_i\}$ of the linearised transformations (19)–(24)

$$\boldsymbol{\mu}' = \hat{\boldsymbol{L}} \cdot \boldsymbol{\mu} \tag{A.1}$$

about the fixed point. It is easily seen that the transformation matrix $\hat{\boldsymbol{L}}$ (A.1) takes the form

$$\hat{L} = \begin{vmatrix} L_{11} & L_{12} & L_{13} & 0 & L_{15} & L_{16} \\ L_{21} & L_{22} & 0 & L_{24} & L_{25} & 0 \\ 0 & 0 & L_{33} & 0 & L_{35} & L_{36} \\ 0 & 0 & 0 & L_{44} & L_{45} & 0 \\ 0 & 0 & L_{53} & L_{54} & L_{55} & L_{56} \\ 0 & 0 & 0 & 0 & 0 & L_{66} \end{vmatrix} \tag{A.2}$$

where the elements $L_{ij}$ can be expressed by the fixed-point values (25)–(27). From (A.2) it follows that the eigenvalue equation

$$\det\{\hat{\boldsymbol{L}} - \lambda \hat{\boldsymbol{I}}\} = 0$$

($\hat{\boldsymbol{I}}$ is the unit matrix) decouples to three equations

$$L_{66} = \lambda \tag{A.3}$$

$$\begin{vmatrix} L_{11}-\lambda & L_{12} \\ L_{21} & L_{22}-\lambda \end{vmatrix} = 0 \tag{A.4}$$

$$\begin{vmatrix} L_{33}-\lambda & 0 & L_{35} \\ 0 & L_{44}-\lambda & L_{45} \\ L_{53} & L_{54} & L_{55}-\lambda \end{vmatrix} = 0. \tag{A.5}$$

The matrix elements entering (A.3)–(A.5) are, from the fixed-point values $u^*$, $v^*$, $w^*$ and $t^*$,

$$\begin{aligned}
l_{11} &= 2 + [18t^* - 4(n+2)u^*]\epsilon \\
l_{22} &= 2 - 4(n+2)\epsilon v^* \\
l_{33} &= \epsilon[1 + 36t^* - 8(n+8)u^*] \\
l_{44} &= \epsilon[1 - 8(m+8)v^*] \\
l_{55} &= \epsilon[1 + 18t^* - 32w^* - 4(n+2)u^* - 4(m+2)v^*] \\
l_{66} &= \epsilon(1 - 2nt^*) \\
l_{12} &= -4m\epsilon\, e^{2s} w^* \qquad l_{21} = -4n\epsilon\, e^{2s} w^* \\
l_{35} &= -8m\epsilon w^* \qquad l_{53} = -4(n+2)\epsilon w^* \\
l_{45} &= -8n\epsilon w^* \qquad l_{54} = -4(m+2)\epsilon w^*
\end{aligned} \tag{A.6}$$

where $L_{ii} = \exp(sl_{ii})$ and $L_{ij} = sl_{ij}$ (for $i \neq j$). Using (A.6) we obtain from (A.3) the expression (29) for $y_t$ and from (A.4) the expression (30) for $y_{r_\psi}$ and $y_{r_\phi}$. Equation (A.5) yields the values of $y_u$, $y_v$, and $y_w$. We shall write equation (A.5) more explicitly only for the interesting coupled case $w^* \neq 0$:

$$y^3 + A\epsilon y^2 + B\epsilon^2 y + C\epsilon^3 = 0 \tag{A.7}$$

where

$$\begin{aligned}
A &= 16(4u^* + 4v^* - w^*) \\
B &= 768u^*w^* + 768v^*w^* + 64(n+8)(m+8)u^*v^* - 64(nm+m+n+8)w^{*2} \\
&\quad - 288(m+8)t^*v^* + 32w^* - 32w^* - 48u^* - 48v^* - 1 \\
C &= 1024(m+8)(n+8)u^*v^*w^* - 256m(m+8)(n+2)v^*w^{*2} \\
&\quad - 256n(n+8)(m+2)u^*w^{*2} + 64(nm+19m+n+44)w^{*2} \\
&\quad - 768[1+6(m+8)t^*]v^*w^* - 768u^*w^* - 16w^*.
\end{aligned} \tag{A.8}$$

To find the coefficients $A$, $B$ and $C$, we use the fact that for $w \neq 0$, $l_{55}$ is simply $l_{55} = -16w^*$ as follows from (27) and (A.6). In the decoupled case $w^* = 0$ we have $y_u = l_{33}$, $y_v = l_{44}$ and $y_w = l_{55}$.

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